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# A note on a problem of heat transport

by

B.R. Damsté

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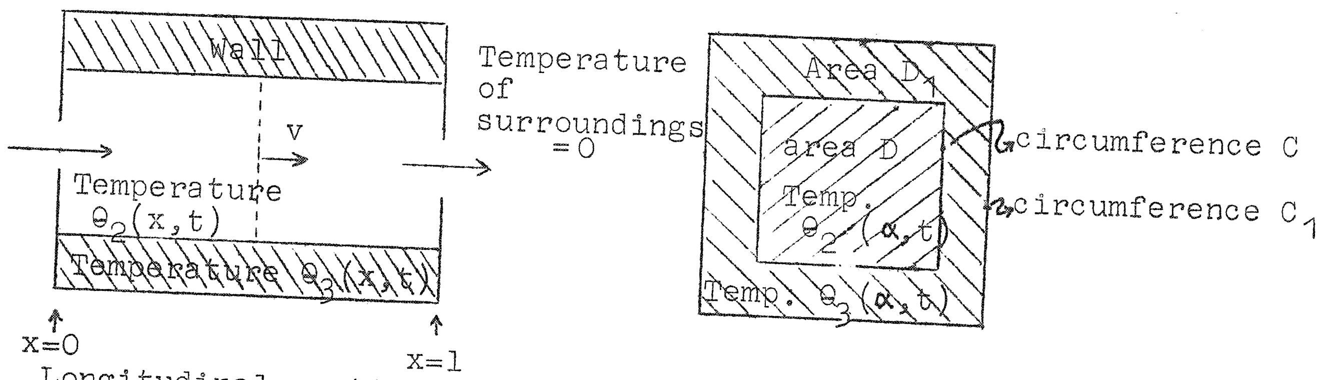
The mathematical model under consideration describes the heating of a railroadcar by hot air, which is blown into it at one end (x=0) and which leaves the car at the opposite end (x=1).**




Longitudinal section

Cross section at $x=\alpha$

For the temperature of the air entering the car, $\theta_1(t)$, we have

(1) $\theta_1(t)$ = arbitrary given function of time t.

(For the particular case $\theta_1(t) = A(1-e^{-\lambda t})$ see (15) seqq.)

The air inside the car is thought of as moving in the x direction only, with a constant speed v. It is assumed that inside the car there is no temperature gradient normal to the x direction, so that for the temperature $\theta_2$ inside the car we have $\theta_2=\theta_2(x,t)$.

For x=0 we have the boundary condition

(2) $$\theta_2(0,t) = \theta_1(t)$$

The car loses heat to the wall, which again gives off heat to the environment.

The temperature inside the wall, $\theta_3$, is assumed to be a function of x and t, $\theta_3=\theta_3(x,t)$, so that inside the wall

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**) For a list of the symbols used see p.7.

there is no temperature gradient normal to the x direction. Furthermore we assume that inside the wall no transport of heat takes place in the x direction. The local loss of heat from the car to the wall is taken to be proportional to $\Theta_2(x,t) - \Theta_3(x,t)$, that from the wall to the surroundings proportional to $\Theta_3(x,t)$.

The proportionality factors are $q_{23}$ and $q_{30}$ respectively per unit of area and per unit of time.

The constant temperature of the surroundings is taken as zero.

For the circumferences C and $C_1$ and the areas D and $D_1$ see the cross section.

The specific heat of air is $q_2$, the specific heat of the wall is $q_3$.

For t=0 we have the initial conditions

(3) $\Theta_2(x,0) = 0$ and $\Theta_3(x,0) = \Theta_{30}(x)$.

We now introduce the constants

(4) $a = \frac{Cq_{23}}{Dq_2}, \quad b = \frac{Cq_{23}}{D_1q_3}, \quad c = \frac{C_1q_{30}}{D_1q_3}$.

This gives us the following simultaneous equations:

$$\begin{cases} \dfrac{\partial \Theta_2(x,t)}{\partial t} + v\,\dfrac{\partial \Theta_2(x,t)}{\partial x} = -a(\Theta_2(x,t)-\Theta_3(x,t)) & (5) \\ \dfrac{\partial \Theta_3(x,t)}{\partial t} = b(\Theta_2(x,t)-\Theta_3(x,t)) - c\Theta_3(x,t). & (6) \end{cases}$$

By applying Laplace transformation to (5) and (6), and using the initial conditions (3), we get the system

$$\begin{cases} p\overline{\Theta}_2(x,p) + v\,\dfrac{\partial \overline{\Theta}_2(x,p)}{\partial x} = -a(\overline{\Theta}_2(x,p)-\overline{\Theta}_3(x,p)) & (5a) \\ p\overline{\Theta}_3(x,p) - \Theta_{30}(x) = b(\overline{\Theta}_2(x,p)-\overline{\Theta}_3(x,p)) - c\overline{\Theta}_3(x,p), & (6a) \end{cases}$$

in which p is the variable of Laplace transformation and the bar indicates the Laplace transform.

Elimination of $\Theta_3$ gives

(7) $$v\,\frac{\partial \overline{\Theta}_2}{\partial x} + \frac{p^2+p(a+b+c)+ac}{p+b+c}\,\overline{\Theta}_2 = \frac{a\,\Theta_{30}}{p+b+c}.$$

The solution of this differential equation is

$$(8)\quad \bar{\Theta}_2(x,p)=\bar{K}(p)\ \exp\left\{-\frac{p^2+p(a+b+c)+ac}{v(p+b+c)}\,x\right\}+\frac{a\Theta_{30}}{p^2+p(a+b+c)+ac}$$

in which $\bar{K}(p)$ is a function of p which has to be determined from the boundary condition (2).

From (2) we see that

$$(9)\quad \bar{\Theta}_2(0,p)=\bar{\Theta}_1(p).$$

From (8) we get

$$(10)\quad \bar{\Theta}_2(0,p)=\bar{K}(p)+\frac{a\Theta_{30}}{p^2+p(a+b+c)+ac}.$$

Denoting the inverse of $\frac{a\Theta_{30}}{p^2+p(a+b+c)+ac}$ by T(t) we have

$$(11)\quad T(t)=\begin{cases} H(t)\dfrac{a\Theta_{30}}{\sqrt{\left(\frac{a+b+c}{2}\right)^2-ac}}\,e^{-\frac{(a+b+c)}{2}t}\,\mathrm{sh}\left(t\sqrt{\left(\frac{a+b+c}{2}\right)^2-ac}\right) & \text{for } (a+b+c)^2>4ac \\ H(t)\dfrac{a\Theta_{30}}{\sqrt{ac-\left(\frac{a+b+c}{2}\right)^2}}\,e^{-\frac{a+b+c}{2}t}\sin\left(t\sqrt{ac-\left(\frac{a+b+c}{2}\right)^2}\right) & \text{for } (a+b+c)^2<4ac \\ H(t)\,a\Theta_{30}\,t\,e^{-\frac{a+b+c}{2}t} & \text{for } (a+b+c)^2=4ac \end{cases}$$

In these formulae H(t) is Heaviside's unit step function

$$H(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$$

We can easily find K(t), the inverse Laplace transform of $\bar{K}(p)$, from (9), (10) and (11) as

$$(12)\quad K(t)=\left\{\Theta_1(t)-T(t)\right\}H(t)$$

For the exponential factor in (8) we have

(13) $\overline{U}(p) \overset{\text{def}}{=} \exp\left\{-\frac{p^2+(a+b+c)p+ac}{v(p+b+c)}x\right\} = e^{-\frac{ax}{v}} \exp\left\{\left(-\frac{p}{v} + \frac{ab/v}{p+b+c}\right)x\right\}.$

Erdélyi et al.: Tables of integral transforms I section 5.5 formula (31) gives:

$$e^{\frac{a}{p}} - 1 \doteqdot \sqrt{\frac{a}{t}}\; I_1(2\sqrt{at}) ,$$

so that the inverse transform of $\overline{U}(p)$ is

(14) $$U(t) = e^{-\frac{ax}{v}}\left\{\delta(t-\frac{x}{v}) + e^{-(b+c)(t-\frac{x}{v})}\sqrt{\frac{abx}{vt-x}}\; I_1\left(2\sqrt{\frac{abx}{v}(t-\frac{x}{v})}\right) H(t-\frac{x}{v})\right\},$$

We thus find the following result for $\theta_2(x,t)$, in which the symbol $*$ denotes convolution:

(15) $$\theta_2(x,t) = K(t) * U(t) + T(t).$$

For $\theta_3(x,t)$ we have from (6) together with the boundary condition (3):

(16) $$\theta_3(x,t) = \theta_{30}\, e^{-(b+c)t} + \frac{b}{b+c}\,\theta_2(x,t).$$

<u>A particular case</u>

We now use the heating function

(17) $$\theta_1(t) = A\,(1-e^{-\lambda t}) ,$$

in which $A$ and $\lambda$ are positive constants.
Then (12) becomes

(18) $$K(t) = \left\{A(1-e^{-\lambda t}) - T(t)\right\} H(t).$$

For $\theta_2(x,t)$ we find from (15), (18) and (14)

(19) $$\theta_2(x,t) = T(t) + e^{-\frac{ax}{v}}\int_0^t \left\{A(1-e^{-\lambda(t-\tau)}) - T(t-\tau)\right\} H(t-\tau)\,.$$

$$.\left\{e^{-(b+c)(\tau-\frac{x}{v})}\sqrt{\frac{abx}{v\tau-x}}\; I_1\left(2\sqrt{\frac{abx}{v}(\tau-\frac{x}{v})}\right)H(\tau-\frac{x}{v}) + \delta(\tau-\frac{x}{v})\right\} d\tau.$$

For $t < \frac{x}{v}$ we have obviously

(20) $\theta_2(x,t) = T(t)$.

Assuming now $t > \frac{x}{v}$, (19) may be reduced to

$$(21)\quad \theta_2(x,t) = T(t) + e^{-\frac{ax}{v}} \left\{ A(1-e^{-\lambda(t-\frac{x}{v})}) - T(t-\frac{x}{v}) \right\} +$$

$$+ Ae^{(-a+b+c)\frac{x}{v}} \int_{x/v}^{t} e^{-\tau(b+c)} \sqrt{\frac{abx}{v\tau-x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau +$$

$$- Ae^{(-a+b+c)\frac{x}{v} - \lambda t} \int_{x/v}^{t} e^{-\tau(b+c-\lambda)} \sqrt{\frac{abx}{v\tau-x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau +$$

$$- e^{(-a+b+c)\frac{x}{v}} \int_{x/v}^{t} e^{-\tau(b+c)}\, T(t-\tau) \sqrt{\frac{abx}{v\tau-x}}\, I_1\left(\frac{2}{v}\sqrt{abx(v\tau-x)}\right) d\tau,$$

which may be simplified to

$$(22)\quad \theta_2(x,t) = T(t) + e^{-\frac{ax}{v}} \left\{ A(1-e^{-\lambda(t-\frac{x}{v})}) - T(t-\frac{x}{v}) \right\} +$$

$$+ Ae^{-\frac{ax}{v}} \int_{0}^{\varphi(x,t)} e^{-\frac{v(b+c)}{4abx} w^2} I_1(w)\, dw +$$

$$- Ae^{-\frac{ax}{v} - \lambda(t-\frac{x}{v})} \int_{0}^{\varphi(x,t)} e^{-\frac{v(b+c-\lambda)}{4abx} w^2} I_1(w)\, dw +$$

$$- e^{-\frac{ax}{v}} \int_{0}^{\varphi(x,t)} e^{-\frac{v(b+c)}{4abx} w^2} T\left(t - \frac{x}{v} - \frac{vw^2}{4abx}\right) I_1(w)\, dw$$

in which $\varphi(x,t) \overset{\text{def}}{=} \frac{2}{v}\sqrt{abx(vt-x)}$.

The temperature $\theta_3(x,t)$ follows from (16).

For $t \to \infty$ the fourth term in the right-hand side of (22) converges even for $\lambda > b+c$ by virtue of the factor $e^{-\lambda t}$ with which the integral is multiplied. The other terms give no difficulties, which means that eventually a steady state is reached.

We are now going to determine the behaviour of the solution $\Theta_2(x,t)$ as $t \to \infty$.

Since a steady state is reached, we may take $\frac{\partial \Theta_2}{\partial t} = \frac{\partial \Theta_3}{\partial t} = 0$ in (5) and (6), which then become

$$\begin{cases} v \frac{\partial \Theta_2}{\partial x} = -a(\Theta_2 - \Theta_3) & (23) \\ b\Theta_2 = (b+c)\Theta_3 . & (24) \end{cases}$$

Elimination of $\Theta_3$ gives

$$v \frac{\partial \Theta_2}{\partial x} = -\frac{ac}{b+c}\Theta_2 \qquad (25)$$

which, together with the boundary condition (2), leads to

$$\Theta_2(x,t) \to A \exp\left\{-\frac{acx}{v(b+c)}\right\} \quad \text{for } t \to \infty . \qquad (26)$$

For the steady-state solution of $\Theta_3(x,t)$ we find from (24) and (26)

$$\Theta_3(x,t) \to \frac{Ab}{b+c} \exp\left\{-\frac{acx}{v(b+c)}\right\} \quad \text{for } t \to \infty . \qquad (27)$$

## List of symbols:

| | |
|---|---|
| $v$: | speed of the air inside the car. |
| $x$: | the variable of place. |
| $t$: | the variable of time. |
| $l$: | length of the car. |
| $p$: | the variable of Laplace transformation. |
| $\Theta_2(x,t)$: | the temperature of the air inside the car. |
| $\Theta_3(x,t)$: | the temperature of the wall. |
| $\Theta_{30}(x)$: | the initial temperature of the wall. |
| $\Theta_1(t)$: | the temperature of the air which is blown into the car at $x=0$. |
| $A, \lambda$ : | constants in the equation of $\Theta_1(t)$. |
| $q_{23}$: | the loss of heat from the inside of the car to the wall is proportional to $q_{23}$ per unit of area and per unit of time. |
| $q_{30}$: | the loss of heat from the wall to the surroundings is proportional to $q_{30}$ per unit of area and per unit of time. |
| $q_2$: | specific heat of air. |
| $q_3$: | specific heat of wall. |
| $C$: | inner circumference of cross section of car wall. |
| $C_1$: | outer " " " " " " " |
| $D$: | area of cross section of car interior. |
| $D_1$: | " " " " " wall. |
| $a$: | $\frac{Cq_{23}}{Dq_2}$, $b$: $\frac{Cq_{23}}{D_1 q_3}$, $c$: $\frac{C_1 q_{30}}{D_1 q_3}$. |
| $H(t)$: | Heaviside's unit step function $H(t)=\begin{cases} 1 \text{ for } t>0 \\ 0 \text{ for } t<0 \end{cases}$ |

The symbol ≑ indicates correspondence in Laplace transformation.

$\left.\begin{matrix}\overline{\Theta}_2(x,p) \\ \overline{\Theta}_3(x,p)\end{matrix}\right\}$: the bar indicates the Laplace transform of the function.